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Heat conduction under a spreading pool

by

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SUMMARY

Calculations are presented of the heat flux into a cold liquid pool spreading on the surface of a semi-infinite solid. Previous one-dimensional conduction models are generalised and results are given for a pool spreading in an arbitrary fashion with arbitrary, time-dependent temperature.

The validity of the one-dimensional approximation is examined in detail. The large-time results of three-dimensional models are shown to depend significantly on the surface boundary condition in a way not encountered in one-dimensional models. In particular a model with specified surface temperature, whilst possibly adequate in the one-dimensional approximation, is seen to be totally inappropriate in a three-dimensional context, yielding an infinite heat flux. Results are presented for finite three-dimensional models using alternative surface conditions. These give estimates of when the one-dimensional model becomes inadequate.

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CONTENTS

1. INTRODUCTION

In order to assess the hazards associated with a spill of toxic or flammable liquid, it is important to know the rate of vaporisation of the liquid. For a boiling cryogenic pool this depends directly on the heat flux into the pool. For a slowly evaporating liquid, the evaporation rate depends on the temperature which, in turn, depends on the heat flux into the pool.

Here we shall examine the heat flux conducted into a cold pool spilt on flat uniform ground. This contribution to the heat flux is in many situations an important one, and in some, for example when cryogens are spilt, the dominant one.

In general the pool may be spreading and its temperature changing with time. This will affect the heat flux into it. Previous analyses[1-6] have not allowed for changing temperature (although the spreading aspect has been treated). Our principal aim, therefore, is to derive a simple estimate of the conducted heat flux allowing for both these effects. This forms a fairly self-contained part of a programme to develop a comprehensive model of vaporising pools.

1.1 The vertical conduction approximation

The analyses of References 1-6 all use the approximation that the heat flux is purely vertical. They consider a circular pool of uniform temperature. In fact the observation that the heat flux is approximately in the vertical direction simplifies the problem enormously. This allows us to derive simple formulae for the heat conducted into a pool of varying temperature. Also, very little need be assumed about the manner in which the pool spreads. The heat flux into the pool is calculated, in this vertical conduction approximation, in Sections 2 and 3 where two different surface boundary conditions, used in References 1 and 5 respectively, are considered. The results of Sections 2 and 3 thus constitute generalisation of the work of References 1 and 5.

1.2 Three-dimensional heat conduction

The one-dimensional heat conduction model is intuitively appealing but eventually, at sufficiently large time, three-dimensional effects will become important. Section 4 is therefore devoted to estimating how large these effects will be, and when they will be important.

In general the boundary condition beneath the pool and that at the surface outside the pool may be different and this makes the three-dimensional conduction model difficult to solve analytically. We therefore consider three simple sets of boundary conditions which allow analytic progress. One of these is shown to be ill-defined (implying an infinite heat flux immediately beneath the edge of the pool) but the other two give an estimate of the size of three-dimensional effects and of how long the one-dimensional model may be considered valid.

1.3 Special functions

Two special functions recur throughout this work. It is therefore worth defining them at the outset.

The first is the Error Function which arises frequently owing to the form of the Green's function of the conduction equation. We shall use the function

$$\mathrm{erfc}(x) \equiv \frac{2}{\sqrt{\pi}} \int_x^{\infty} d\xi \; e^{-\xi^2} \qquad \ldots (1.1)$$

The second relevant function is the Bessel function, which arises in problems with cylindrical symmetry. Here we shall need the modified Bessel function of the first kind which may conveniently be defined by

$$I_n(x) = \frac{1}{\pi} \int_0^{\pi} d\theta \; e^{x \cos\theta} \cos(n\theta) \qquad \text{... (1.2)}$$

The notation adopted for these functions is exactly as in Reference 7, where their properties are given in detail. It will also be convenient to define

$$\overline{\mathrm{erfc}}(x) = e^{x^2} \mathrm{erfc}(x) \qquad \text{... (1.3)}$$

and

$$\tilde{I}_n(x) = e^{-x} I_n(x) \qquad \text{... (1.4)}$$

These functions are not defined explicitly as such in Reference 7, but have the merit that they are much more smoothly dependent on their arguments than are erfc and I_n. Their properties are readily derived from those of erfc and I_n.

Various integrals involving these functions occur in this work. Reference 7 is again indispensable. The table of Laplace transforms is particularly helpful in this respect.

1.4 Summary

The remainder of this Report is organised as follows. Vertical conduction models are presented in Sections 2 and 3. Three-dimensional conduction effects are considered in Section 4. The results are summarised in detail in Section 5.

2. SOLUTION OF THE ONE-DIMENSIONAL CONDUCTION EQUATION WITH A SPECIFIED SURFACE TEMPERATURE

2.1 The vertical conduction problem

Consider a cold pool of liquid spreading on a horizontal solid surface. We shall take the solid to be a semi-infinite body described by a depth co-ordinate $z \geqslant 0$ and a horizontal 2-vector co-ordinate $\vec{x} = (x_1, x_2)$. If the pool is sufficiently large, then, at least for a finite period of time, the heat flux in the solid beneath the pool will be primarily in the vertical direction. In this approximation the temperature field $T(t, z, \vec{x})$ satisfies

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial z^2} \qquad \text{... (2.1)}$$

where α is the thermal diffusivity of the solid.

We shall assume that the solid is initially at uniform temperature and choose this to be the zero of the temperature scale. Thus the initial condition is

$$T(0, z, \vec{x}) = 0 \qquad \text{... (2.2)}$$

The boundary condition at the surface will depend on how the pool spreads, how its temperature varies, and on how heat is transferred from the pool to the ground. If the pool is in perfect thermal contact with the ground then the surface temperature under the pool will be that of the pool. The surface temperature outside the pool will be the ambient air temperature which we take to be the same as the initial ground temperature.

Let us describe the area covered by the pool by the set of points $\vec{x} \in P(t)$. (Later we shall consider a concentrically spreading circular pool of radius $R(t)$ for which the relevant set consists of $\vec{x}$ such that $\vec{x} \cdot \vec{x} \leqslant R^2(t)$, but for the moment there is no advantage to be gained by examining this particular case.)

The boundary condition is then

$$T(t, 0, \vec{x}) = \Phi(t, \vec{x}) \qquad \ldots (2.3)$$

where Φ is the pool temperature for $\vec{x} \in P(t)$ and is zero elsewhere.

2.2 The temperature field

The solution to the problem defined by (2.1)-(2.2) is

$$T(t, z, \vec{x}) = \int_0^t d\tau \, G_v(z, t-\tau) \, \Phi(\tau, \vec{x}) \qquad \ldots (2.4)$$

where the Green's function G_v is given by

$$G_v(z, t) = \frac{z}{(4\pi\alpha t^3)^{1/2}} \exp\left(\frac{-z^2}{4\alpha t}\right) \qquad \ldots (2.5)$$

(This is readily verified by Laplace transforming on the variable t.) Note that the vertical conduction assumption means that $T = 0$ for all points which have never been under the pool, that is for all $(\vec{x}, t)$ such that

$$T(t, z, \vec{x}) = 0 \qquad \text{for } \vec{x} \notin \bigcup_{\tau\in[0,t]} [P(t)] \qquad \ldots (2.6)$$

2.3 The heat flux

The vertical heat flux density (defined towards the surface in the negative z-direction) is

$$q(t, z, \vec{x}) = K \frac{\partial T}{\partial z} \qquad \ldots (2.7)$$

where K is the thermal conductivity of the solid. From (2.4) we find

$$q(t, z, \vec{x}) = K \int_0^t d\tau \, G_v'(z, t-\tau) \, \Phi(\tau, \vec{x}) \qquad \ldots (2.8)$$

where

$$G_v'(z, t) = \frac{\partial}{\partial z} G_v(z, t) \qquad \ldots (2.9)$$

This integral is simplified in Appendix A where, in particular, it is shown that the heat conducted at the surface is

$$q(t, 0, \vec{x}) = \frac{-K\Phi(t, \vec{x})}{\sqrt{\pi t}} - \frac{K}{\sqrt{4\pi\alpha}} \int_0^t d\tau \, (t-\tau)^{-3/2} \left[\Phi(t, \vec{x}) - \Phi(\tau, \vec{x})\right] \qquad \ldots (2.10)$$

The total heat flux into the base of the pool is

$$Q_T(t) = \int_{P(t)} d^2\vec{x} \; q(t, 0, \vec{x}) \qquad \ldots (2.11)$$

This depends on the manner in which the pool spreads.

Let the area of the pool at time t be A(t), and define the area-averaged pool temperature $\phi(t)$ by

$$\phi(t)\,A(t) \;=\; \int_{P(t)} d^2\vec{x}\;\Phi\,(t,\vec{x}) \qquad \ldots (2.12)$$

Furthermore, let us define a function $\tilde{A}(t,\tau)$ by

$$\phi(\tau)\,\tilde{A}(t,\tau) \;\equiv\; \int_{P(t)} d^2\vec{x}\,\Phi\,(\tau,\vec{x})$$

$$= \int_{P(t)\cap P(\tau)} d^2\vec{x}\,\Phi\,(\tau,\vec{x}) \qquad \ldots (2.13)$$

Then the total heat flux into the base of the pool becomes

$$Q_T(t) \;=\; \frac{-K\phi(t)A(t)}{\sqrt{\pi\alpha t}} \;-\; \frac{K}{\sqrt{4\pi\alpha}}\int_0^t d\tau\,(t-\tau)^{-3/2}\left[\phi(t)\,A(t) - \phi(\tau)\,\tilde{A}(t,\tau)\right] \qquad \ldots (2.14)$$

This result applies to a pool of arbitrary temperature $\Phi\,(t,\vec{x})$ spreading in an arbitrary manner.

It simplifies considerably if the pool temperature is uniform $(\Phi\,(t,\vec{x}) = \phi(t)$ for $\vec{x}\in P(t))$ and if the pool spreads monotonically outwards (such that any point covered by the pool at time τ is covered for all $t \geqslant \tau$; or P(t) contains $P(\tau)$ for $\tau\in[0,t]$). In this case $\tilde{A}(t,\tau) \rightarrow A(\tau)$ and (2.14) becomes

$$Q_T(t) \;=\; \frac{-K\phi(t)A(t)}{\sqrt{\pi\alpha t}} \;-\; \frac{K}{\sqrt{4\pi\alpha}}\int_0^t d\tau\,(t-\tau)^{-3/2}\left[\phi(t)\,A(t) - \phi(\tau)\,A(\tau)\right] \qquad \ldots (2.15)$$

Let us note for future reference that the simplest case is that of a pool boiling in a bund. In this case the temperature and area of the pool are both constant and the heat flux reduces to

$$Q_T(t) \;=\; -\,\frac{K\phi A}{(\pi\alpha t)^{1/2}} \qquad \ldots (2.16)$$

The solution (2.14) with the simpler special cases (2.15) and (2.16) constitute a generalisation of the results of Shaw and Briscoe[1] to a pool with an arbitrary time-dependent temperature, and spreading in an arbitrary fashion.

The $t^{-1/2}$ singularity in (2.16) is a consequence of the singular nature of the boundary condition at $t = 0$, and may be expected quite generally from (2.14). It does not, however, cause problems here because the total heat transfer to the pool up to time t vanishes as $t \rightarrow 0$. Nevertheless, a non-singular boundary condition will give a physically more acceptable one-dimensional conduction model (which we derive in Section 3) and, as we shall see, is crucial if three-dimensional effects are to be considered.

3. SOLUTION OF THE ONE-DIMENSIONAL CONDUCTION EQUATION WITH A FIXED HEAT TRANSFER COEFFICIENT AT THE SURFACE

3.1 The boundary condition

An alternative boundary condition is that the surface of the ground is not at the pool temperature but that the heat flux into the pool is proportional to the temperature difference between the ground and the pool. That is

$$q(t, 0, \vec{x}) = \tilde{h}\left[T(t, 0, \vec{x}) - \Phi(t, \vec{x})\right] \quad \dots (3.1)$$

where $\tilde{h}$ is a constant heat transfer coefficient.

This boundary condition has been used in the context of boiling cryogenic pools by Welker and Cavin.[5]

3.2 Solution of the model

The one-dimensional conduction model with this boundary condition is readily solved in terms of the field

$$S(t, z, \vec{x}) \equiv T(t, z, \vec{x}) - \frac{1}{h}\frac{\partial T}{\partial z} \quad \dots (3.2)$$

where

$$h \equiv \tilde{h}/K \quad \dots (3.3)$$

Now S obeys the same conduction equation as T, and satisfies the boundary condition

$$S(t, 0, \vec{x}) = \Phi(t, \vec{x}) \quad \dots (3.4)$$

The solution, therefore, is exactly analogous to that in Section 2.2:

$$S(t, z, \vec{x}) = \int_0^t d\tau\, G_V(z, t-\tau)\, \Phi(\tau, \vec{x}) \quad \dots (3.5)$$

Now (3.2) can be inverted to give

$$T(t, z, \tau) = he^{hz} \int_z^\infty dz'\, e^{-hz'}\, S(t, z', \vec{x}) \quad \dots (3.6)$$

and the vertical heat flux is

$$q(t, z, \vec{x}) = Kh[\dot{T} - S] \quad \dots (3.7)$$

Equation (3.5) therefore implies

$$T(t, z, \vec{x}) = \int_0^t d\tau\, G_1(z, t-\tau)\, \Phi(\tau, \vec{x}) \quad \dots (3.8)$$

and

$$q(t, z, \vec{x}) = K \int_0^t d\tau\, G_2(z, t-\tau)\, \Phi(\tau, \vec{x}) \quad \dots (3.9)$$

where G_1 and G_2 are found straightforwardly from G_V. These are given in Appendix B.

3.3 The heat flux at the surface

The heat flux at the surface may now be found following the method of Appendix A. In particular using the identity (A.2) we find

$$q(t, 0, \vec{x}) = K\Phi(t, \vec{x}) \lim_{z \to 0} \int_0^t d\tau\, G_2(z, t-\tau)$$

$$- K \int_0^t d\tau\, G_2(0, t-\tau)\left[\Phi(t, \vec{x}) - \Phi(\tau, \vec{x})\right] \quad \ldots (3.10)$$

or

$$q(t, 0, \vec{x}) = \frac{-K\Phi(t, \vec{x})}{(\alpha t_0)^{1/2}} \overline{\text{erfc}} \left(\left(\frac{t}{t_0}\right)^{1/2}\right)$$

$$- K \int_0^t d\tau\, G_2(0, t-\tau)\left[\Phi(t, \vec{x}) - \Phi(\tau, \vec{x})\right] \quad \ldots (3.11)$$

where $\overline{\text{erfc}}$ is defined in Section 1.4, and the timescale t_0 is defined by

$$t_0 = (h^2\alpha)^{-1} \quad \ldots (3.12)$$

and

$$G_2(0, t-\tau) = \frac{1}{(\alpha t_0^3)^{1/2}} \left\{ \frac{1}{\sqrt{\pi}} \left(\frac{t_0}{t-\tau}\right)^{1/2} - \overline{\text{erfc}} \left(\left(\frac{t-\tau}{t_0}\right)^{1/2}\right)\right\} \quad \ldots (3.13)$$

as given in Appendix B.

The total heat flux into the pool is now found exactly as in Section 2.3. It is

$$Q_T(t) = \frac{-K}{(\alpha t_0)^{1/2}} \phi(t) A(t)\, \overline{\text{erfc}} \left(\left(\frac{t}{t_0}\right)^{1/2}\right)$$

$$- K \int_0^t d\tau\, G_2(0, t-\tau)\left[\phi(t) A(t) - \phi(\tau) \tilde{A}(t, \tau)\right] \quad \ldots (3.14)$$

where $\tilde{A}(t, \tau)$ is defined in Equation (2.13). For a pool or uniform temperature spreading monotonically outwards, $\tilde{A}(t, \tau)$ reduces to $A(\tau)$, the area at time τ.

For a circular pool of constant area A and temperature ϕ, then (3.14) reduces to

$$Q_T(t) = \frac{-K}{(\alpha t_0)^{1/2}} \phi A\, \overline{\text{erfc}} \left(\left(\frac{t}{t_0}\right)^{1/2}\right) \quad \ldots (3.15)$$

as given in Reference 5.

The limit of perfect thermal contact is $h \to \infty$. In this limit the boundary condition (3.1) reduces to that discussed earlier. As $h \to \infty$ then $t_0 \to 0$ and the asymptotic properties[7] of the Error Function mean that the results of this section reduce to those of Section 2.3. (For example (3.14) → (2.14).)

The length scale h^{-1} determines the distance over which the effect of the boundary condition is felt. It is possibly more convenient, however, to think in terms of the time-scale t_0 defined in (3.12). For $t \gg t_0$ there is effectively no difference in the heat flux found in this section and that of Section 2.3. For $t \ll t_0$ the effect of the imperfect thermal contact is important. Welker and Cavin[5] find that t_0 is of the order of a few minutes for LPG on concrete.

The results of this section constitute a generalisation of the model given by Welker and Cavin[5] to include the possibility of a spreading pool of varying temperature, just as the results of the previous section generalise those of Shaw and Briscoe.[1]

4. THREE-DIMENSIONAL CONDUCTION EFFECTS

4.1 Introduction

For many applications the simple one-dimensional conduction models presented in Sections 2 and 3 will be adequate. To be confident that this is the case, however, means that some investigation is needed of the size of 'correction terms' due to horizontal conduction effects. The three-dimensional conduction equation brings forth some problems not encountered in the one-dimensional model.

For example, in the one-dimensional model, the boundary condition outside the pool is only important if the pool spreads to cover a certain point, then uncovers it again, and subsequently encounters that point for a second time. To see this consider a surface point $(\vec{x}, 0)$ which is initially outside the pool. Points $(\vec{x}, z)$ will remain at the initial, uniform, ambient temperature T = 0 until such time as the pool front reaches $(\vec{x}, 0)$. This will be true in the vertical conduction approximation independent of the boundary condition outside the pool. After the pool front reaches $(\vec{x}, 0)$ the temperature at $(\vec{x}, z)$ will depend on the boundary condition within the pool. Only if the point $(\vec{x}, 0)$ is uncovered again can the temperature at (x,z) be affected by the boundary condition outside the pool. And then only if $(\vec{x}, 0)$ is reached by the pool again can this affect the heat flow into the pool.

Of course as soon as the possibility of horizontal conduction is admitted this argument becomes invalid. Heat may be transferred from the air to the ground outside the pool, through the ground and into the pool. The boundary condition outside the pool is therefore important.

We shall consider three possibilities which allow analytic progress:

(i) that the pool and the air are both in perfect thermal contact with the ground,

(ii) that the pool and the air are both in imperfect thermal contact with the ground with the **same** heat transfer coefficient,

(iii) that the pool is in perfect thermal contact with the ground but that no heat is transferred from the air to the ground.

The conditions (i) and (iii) provide two extreme possibilities for generalising the results of Section 2. The condition (ii) provides a very particular generalisation of the model of Section 3.

None of these models is ideal. We shall show that the first is ill-defined. In the second model one may doubt the initial assumption that the heat transfer coefficients are the same. The mixed boundary condition in the third model makes solution more difficult and results are presented for the time-independent solution only. Nevertheless, the models taken together do provide an indication of when the vertical conduction model is likely to be inadequate, and as they are the simplest three-dimensional generalisations of the models of Sections 2 and 3, they merit examination.

We shall consider, then, the solution of the three-dimensional conduction equation

$$\frac{\partial T}{\partial t} = \alpha \left\{ \frac{\partial^2 T}{\partial z^2} + \vec{\nabla} . \vec{\nabla} T \right\} \qquad \dots (4.1)$$

with various boundary conditions. Let us note first that the solution for specified surface temperature

$$T(t, 0, \vec{x}) = \Phi(t, \vec{x}) \qquad \dots (4.2)$$

is

$$T(t, z, \vec{x}) = \int_0^t d\tau \int d^2\vec{y}\; G_V(z, t-\tau)\, G_H(\vec{x}-\vec{y}, t-\tau)\, \Phi(\tau, \vec{y}) \qquad \dots (4.3)$$

where G_V is as given in (2.5) and G_H is the 'horizontal conduction' Green's function given by

$$G_H(\vec{x}, t) = \frac{1}{4\pi\alpha t} \exp\left(-\frac{x^2}{4\alpha t}\right) \quad \ldots (4.4)$$

The upward vertical heat flux is

$$q(t, z, \vec{x}) = K \int_0^t d\tau \int d^2\vec{y}\; G_V'(z, t-\tau)\, G_H(\vec{x}-\vec{y}, t-\tau)\, \Phi(\tau, \vec{y}) \quad \ldots (4.5)$$

where G_V' is given in Equation (B.3). This solution is most easily verified by Laplace transforming on t and Fourier transforming on $\vec{x}$.

4.2 Perfect thermal contact at the surface

The solution (4.3) is directly applicable to the case where there is perfect thermal contact at the surface. However, if the surface temperature is discontinuous then the problem is ill-defined. To see this consider a time-independent surface temperature $\Phi(x)$ defined by

$$\left.\begin{aligned} \Phi(\vec{x}) &= \phi\,,\ x_1 \leqslant 0 \\ &= 0\,,\ x_1 \geqslant 0 \end{aligned}\right\} \quad (4.6)$$

(This may be thought of as representing an infinite pool covering exactly half the ground surface, but it serves to illustrate the problems at the edge.)

In this case, we note from (4.4) that

$$\int_{-\infty}^{0} dy_1 \int_{-\infty}^{\infty} dy_2\, G_H(\vec{x}-\vec{y}, t-\tau) = \frac{1}{2}\, \text{erfc}\left(\frac{x_1}{(4\alpha(t-\tau))^{1/2}}\right) \quad \ldots (4.7)$$

and so, from (4.5) the upward vertical heat flux becomes

$$q(t, z, \vec{x}) = \frac{K\phi}{2} \int_0^t d\tau\, G_V'(z, t-\tau)\ \text{erfc}\left(\frac{x_1}{(4\alpha(t-\tau))^{1/2}}\right) \quad \ldots (4.8)$$

Using Equation (B.3) for G_V', this integral can be done by parts to yield

$$q(t, z, \vec{x}) = K\phi \left[\frac{-1}{(4\pi\alpha t)^{1/2}}\, e^{-\frac{z^2}{4\alpha t}}\ \text{erfc}\left(\frac{x_1}{(4\alpha t)^{1/2}}\right) + \frac{x_1}{\pi(x_1^2+z^2)}\, e^{-\left(\frac{x_1^2+z^2}{4\alpha t}\right)}\right] \quad \ldots (4.9)$$

This is sufficiently singular at $x_1 = z = 0$ that the total heat flux at the surface ($z = 0$) into an infinitessimal area $-\varepsilon < x_1 < 0$ at the edge of the 'pool' is infinite (see Fig. 1).

This problem occurs for any model where both the pool and the air are in perfect thermal contact with the ground and where the surface temperature is discontinuous. We must therefore reject this as a generalisation of the model of Section 2.

4.3 Uniformly imperfect thermal contact at the surface

4.3.1 General results

It turns out that the simplest, analytically tractable three-dimensional model is the generalisation of the model of Section 3, where the pool and the air are assumed to have the same heat transfer coefficient with the ground. In this case one defines the field $S(t, z, \vec{x})$ as in Section 3 and finds

$$S(t, z, \vec{x}) = \int_0^t d\tau \int d^2\vec{y}\; G_V(z, t-\tau)\, G_H(\vec{x}-\vec{y}, t-\tau)\, \Phi(\tau, \vec{y}) \qquad \ldots (4.10)$$

with the temperature

$$T(t, z, \vec{x}) = \int_0^t dt \int d^2\vec{y}\; G_1(z, t-\tau)\, G_H(\vec{x}-\vec{y}, t-\tau)\, \Phi(\tau, \vec{y}) \qquad \ldots (4.11)$$

and the vertical heat flux

$$q(t, z, \vec{x}) = K \int_0^t d\tau \int d^2\vec{y}\; G_2(z, t-\tau)\, G_H(\vec{x}-\vec{y}, t-\tau)\, \Phi(\tau, \vec{y}) \qquad \ldots (4.12)$$

where the various Green's functions are as previously defined (see Appendix B).

The heat flux at the surface is

$$q(t, 0, \vec{x}) = K \int_0^t d\tau\, G_2(0, t-\tau) \int d^2\vec{y}\; G_H(\vec{x}-\vec{y}, t-\tau)\, \Phi(\tau, \vec{y}) \qquad \ldots (4.13)$$

4.3.2 *Results for a pool of uniform temperature*

The total heat flux into the pool

$$Q_T(t) = \int_{P(t)} d^2\vec{x}\; q(t, 0, \vec{x}) \qquad \ldots (4.14)$$

can only be simplified if the temperature distribution in the pool is known. Let us therefore look at a pool of uniform temperature:

$$\begin{aligned} \Phi(t, \vec{x}) &= \phi(t) \quad ; \quad \vec{x} \epsilon P(t) \\ &= 0 \quad ; \quad \vec{x} \epsilon P(t) \end{aligned} \qquad \ldots (4.15)$$

In this case we obtain

$$Q_T(t) = K \int_0^t d\tau\, G_2(0, t-\tau)\, C(t, \tau)\, \phi(\tau) \qquad \ldots (4.16)$$

where the 'horizontal conduction factor' C is given by

$$C(t, \tau) \equiv \int_{P(t)} d^2x \int_{P(\tau)} d^2y\; G_H(\vec{x}-\vec{y}, t-\tau) \qquad \ldots (4.17)$$

This factor depends on the shape of the pool and (4.16) cannot be simplified further without specifying the shape. For a concentrically spreading circular pool C simplifies somewhat as is shown in Appendix C. The simplest form we have found is given in Equations (C.17) and (C.16). For a pool in a circular bund a simpler expression for C, denoted by $C_B(t-\tau)$, is given in Equation (C.21). $G_2(0, t-\tau)$ is given in Equation (B.8).

The form of C given in (C.17) and (C.16) means that two integrals must be done in general to evaluate the heat flux (4.16) into the pool. However, if the pool is not spreading, the simpler form (C.21) for $C_B(t-\tau)$ means that only one integral (that in (4.16)) remains.

4.3.3 *Results for a pool of constant temperature and area*

If the area and temperature of the pool are both constant (for example for a pool boiling in a bund) then (4.16) can be simplified further. Using (B.8) and (C.22) and integrating (4.16) by parts we find for this case

$$Q_T(t) = -\frac{(\pi R^2)K\phi}{\sqrt{\alpha t_0}}\left[\overline{\text{erfc}}\left(\left(\frac{t}{t_0}\right)^{1/2}\right)\left\{1 - \tilde{I}_0\left(\frac{t_H}{t}\right) - \tilde{I}_1\left(\frac{t_H}{t}\right)\right\} + \int_1^{\infty}\frac{dx}{x}\,\overline{\text{erfc}}\left(\left(\frac{t}{xt_0}\right)^{1/2}\right)\tilde{I}_1\left(\frac{xt_H}{t}\right)\right] \qquad \ldots (4.18)$$

where the functions $\overline{\text{erfc}}$ and $\tilde{I}_n$ are defined in Section 1.4. The time-scales t_0, t_H are associated with the surface boundary condition:

$$t_0 \equiv (h^2\alpha)^{-1} \qquad \ldots (4.19)$$

and with the effect of horizontal conduction:

$$t_H \equiv \frac{R^2}{4\alpha} \qquad \ldots (4.20)$$

The vertical conduction approximation is recovered in the limit $t_H \to \infty$. The model with perfect thermal contact at the surface corresponds to $t_0 \to 0$. But, as we have seen, this limit only gives a finite value of $Q_T(t)$ if the limit $t_H \to \infty$ is taken first.

For a sufficiently large pool in sufficiently good thermal contact with the ground, then t_H will be much greater than t_0. In this case we may investigate three simpler régimes: $t \ll t_0$, $t_0 \ll t \ll t_H$ and $t_H \ll t$.

Small t:

For $t_0 \ll t_H$ and $t \ll t_H$ the expression (4.18) may be approximated by

$$\frac{Q_T(t)}{\pi R^2} \approx \frac{-K\phi}{\sqrt{\alpha t_0}}\,\overline{\text{erfc}}\left(\left(\frac{t}{t_0}\right)^{1/2}\right) \qquad \ldots (4.21)$$

which for $t \ll t_0$ may be expanded as

$$\frac{Q_T(t)}{\pi R^2} \approx \frac{-K\phi}{\sqrt{\alpha t_0}}\left\{1 - \frac{2}{\sqrt{\pi}}\left(\frac{t}{t_0}\right)^{1/2} + O\left(\frac{t}{t_0}\right)\right\} \qquad \ldots (4.22)$$

The approximation (4.21) is just the vertical conduction result (3.15).

Intermediate t:

The régime $t_0 \ll T \ll t_H$ is particularly interesting, for the vertical conduction model suggests that the heat flux should become independent of the heat transfer coefficient at the surface. Furthermore for $t \ll t_H$ horizontal conduction effects should not be important. In fact the approximation $t \ll t_H$ is already incorporated in (4.21) which for $t \gg t_0$ yields

$$\frac{Q_T(t)}{\pi R^2} \simeq -\frac{K\phi}{(\pi\alpha t)^{1/2}} \qquad \ldots (4.23)$$

A better approximation in this régime can be found by expanding (4.18) simultaneously in powers of t/t_0 and t_H/t. This yields

$$\frac{Q_T(t)}{\pi R^2} = \frac{K\phi}{(\pi\alpha t)^{1/2}}\,\gamma(t) \qquad \dots (4.24)$$

with

$$\gamma(t) = \left[1 - \frac{1}{2}\frac{t_0}{t} + \dots\right]\left[1 - \left(\frac{t}{2\pi t_H}\right)^{1/2} + \dots\right] +$$

$$\left(\frac{t}{2\pi t_H}\right)^{1/2}\left\{1n\,\frac{t}{t_0} + 0(1) + \frac{3}{8}\frac{t_0}{t_H}\left(\frac{t}{t_0} - \frac{1}{2}\ln\frac{t}{t_0} + 0(1)\right)\right.$$

$$\left. - \frac{15}{128}\left(\frac{t_0}{t_H}\right)^2\left(\frac{1}{2}\left(\frac{t}{t_0}\right)^2 - \frac{t}{t_0} + \frac{3}{4}\ln\left(\frac{t}{t_0}\right) + 0(1)\right) + \dots\right\} \qquad \dots (4.25)$$

This somewhat cumbersome expansion catalogues the leading correction terms to (4.23). It illustrates the problem of the limit, $t_0 \rightarrow 0$, of perfect thermal contact. The leading logarithmic term diverges in this limit, unless $t_H \rightarrow \infty$ is taken first; that is unless purely vertical conduction is considered. This term also means that the condition $t_0 \ll t \ll t_H$ for (4.23) to be valid must be augmented by the condition

$$\left(\frac{t}{t_H}\right)^{1/2}\ln\left(\frac{t}{t_0}\right) \ll 1 \qquad \dots (4.26)$$

Large t

In the limit $t \rightarrow \infty$, $Q_T(t)$ tends to a constant. The dominant contribution to the integral in (4.18) comes from the region $0\,(t/t_0) \geqslant x \geqslant 0\,(t/t_H)$. An asymptotic expansion of the integrand in this region leads to

$$\frac{Q_T(t)}{\pi R^2} \underset{t\rightarrow\infty}{\rightarrow} \frac{-K\phi}{\pi(2\alpha t_H)^{1/2}}\left\{\ln\left(\frac{t_H}{t_0}\right) + c\right\} \qquad \dots (4.27)$$

where c is a constant, which this method does not easily specify. However, comparing (4.27) with a numerical integration of (4.18) for various large values of t/t_H and of t_H/t_0 gives $c = 2\cdot00$ to three significant figures. In this régime horizontal conduction is of paramount importance. The heat transfer coefficient at the surface is also significant as t_0 appears prominently in (4.27). This is because horizontal conduction effects mean that the surface boundary conditions outside the pool are important.

To illustrate the effect of horizontal conduction, let us consider circular pools of propane of area $0\cdot47\ m^2$ ($5\ ft^2$), $47\ m^2$ and $4{,}700\ m^2$ on perlite concrete. For the first example, Welker and Cavin[5] find *$K = 0\cdot94$ Btu/hrft°F ($1\cdot63$ J/sm°K), $\alpha = 0\cdot0472\ ft^2/hr$ ($1\cdot22\ 10^{-6}\ m^2/s$), and $\tilde{h} = 20$ Btu/hr ft^2 °F (114 J/sm^2 °K). These values give $h = 1\cdot4$ cm, $t_0 = 168$ s, and $t_H = 3\cdot03\ 10^4$ s, $3\cdot03\ 10^6$ s, and $3\cdot03\ 10^8$ s for the three different pool areas. Even for the small pool t_H/t_0 is as large as 180 but, as can be seen from Fig. 2, this is not large enough to exhibit the central régime discussed above. This régime is clearly exhibited by the largest of the three pools considered.

In fact the heat transfer coefficient of $114\ Js^{-1}\,m^{-2}\,K^{-1}$ found in Reference 5 for the contact between the pool and the ground is rather larger than one would expect for forced convection

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* *K is found by fitting evaporation data to the theory of Section 2.4 and **not** by any independent method.*

heat transfer between ground and air, for example, as given in Reference 8.* Thus the asymptotic heat flux which, we have argued, depends on the heat transfer coefficient between the ground and the air outside the pool is overestimated.

4.4 Other boundary conditions

A wider generalisation of the model of Section 3 would be to take a heat transfer coefficient $\tilde{h}_1$ beneath the pool and a different heat transfer coefficient $\tilde{h}_2$ where the air is in contact with the ground.

Our results thus far show that the case $\tilde{h}_1 = \tilde{h}_2 \rightarrow \infty$ is ill-defined and that $\vec{h}_1 = \vec{h}_2$ = finite is soluble for a pool of uniform temperature, but fairly complicated.

One other situation yields readily to analysis; that where $\vec{h}_1 \rightarrow 0$. In this case, however, as far as we know, results are only available for steady conduction into a pool of uniform temperature. We see from the results of Section 4.3 that a steady situation may be approached when $t \gg t_H = R^2/4\alpha$ and so the result for this case may be expected to be applicable in this limit. The temperature field (see Reference 9, Section 8.2) is

$$T = \frac{2\phi}{\pi} \sin^{-1} \left\{ \frac{2R}{\left[(r-R)^2 + z^2\right]^{1/2} + \left[(r+R)^2 + z^2\right]^{1/2}} \right\} \qquad \ldots (4.28)$$

where

$$r = \left| \vec{x} \right| .$$

This implies a heat flux at large time given by

$$\frac{Q_T}{\pi R^2} = \frac{-K\phi}{\pi (2\alpha t_H)^{1/2}} \cdot 2\sqrt{2} \qquad \ldots (4.29)$$

which can be compared directly with (4.27). The comparison is illustrated in Fig. 3. In fact, owing to heat transfer from the air to the ground outside the pool, the heat flux at large time may be expected to be larger than (4.29) but not as large as (4.27) for most applications.

If it turns out, in any given application, that the one-dimensional approximation is inadequate and that the model of Section 4.3 grossly overestimates the heat flux into the pool, then it may prove better to use a one-dimensional model until the predicted heat flux drops down to (4.29) and thereafter use (4.29). This method provides a simple pragmatic approach to estimating horizontal conduction effects, and forms the basis of the method we propose to use for modelling spreading, vaporising pools.

5. SUMMARY AND CONCLUSIONS

5.1 Summary

The aim of this work is to calculate the total heat flux into a cold pool of arbitrary, time-dependent temperature spreading in an arbitrary fashion on smooth, flat, uniform ground.

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* *Brighton[8] gives a model for the mass flux of vapour from an evaporating pool. It is easily adapted to give the heat flux from the air to a cold area of ground. The result is $\tilde{h} = c_p \rho_a u_* j$ where c_p and ρ_a are the specific heat and density of air, u_* is the atmospheric friction velocity, and j is a dimensionless factor dependent on the Prandtl number of air and the aerodynamic roughness of the ground, but which depends only very weakly on the dimensions of the cold spot and on u_*. For the situations discussed here, we expect j to be roughly 0·05 to 0·15 and $c_p \rho_a \sim 1200\ Jm^{-3}K^{-1}$. Thus $\tilde{h} \ll 114\ Js^{-1}m^{-1}K^{-1}$ even for a fairly turbulent atmosphere.*

This is achieved in the approximation where heat is assumed to be conducted purely vertically and the results are given in Equation (2.14) for a pool in perfect thermal contact with the ground, and in (3.14) if there is a constant, finite, heat transfer coefficient between the pool and the ground. In both cases the results are particularly simple of spatial variation of temperature is neglected, and if the pool spreads outwards in such a way that a point on the ground once covered is not re-exposed. (The temperature need not be assumed constant in time and the pool need not be assumed circular.)

Section 4 is devoted to an examination of the validity of the vertical conduction approximation. Results are presented for the three-dimensional conduction problem with three possible surface boundary conditions:

(i) if the pool and the surrounding air are both in perfect thermal contact with the ground and are initially at different temperatures, the problem is shown to be ill-defined (see Section 4.2). The pathological nature of the boundary condition which maintains a temperature discontinuity does not seem to be widely appreciated. For example, the problem defined in Section 4.2 is tackled in Reference 10, where these boundary conditions are stated explicitly. However, the approximate solution given there to the three-dimensional problem is finite and corresponds more to the boundary conditions of Section 4.4 than to any others.

(ii) if the heat transfer coefficients between the pool and the ground and between the surrounding air and the ground are constant, finite, and equal, then an expression for the heat flux into the pool may be readily obtained (see Section 4.3). The result is given in Equation (4.13). The total heat flux into the pool is given by a five-fold integral. Three of these may be done if the pool is circular and if the spatial variation of its temperature is neglected. The result for this case is given in Equations (4.16), (C.17) and (C.16). This double-integral form is only slightly less readily usable than the single integral found for the one-dimensional case. We have reduced the three-dimensional model to a single integral for the case of a pool at constant temperature which is not spreading. The result is compared with the corresponding one-dimensional conduction model to give an estimate of the effect of neglecting horizontal conduction.

(iii) if the pool is in perfect contact with the ground and no heat is transferred from the air to the ground, then the steady conduction problem is soluble for a pool of constant area and constant, uniform temperature. The result is quoted (in Section 4.4) from the literature and compared with the corresponding ones from boundary condition (ii) and from the one-dimensional model.

The results of the three-dimensional conduction models indicate that the simpler one-dimensional conduction models are valid until times of order aA/α, where 'a' is a constant, A is the pool area, and α is the thermal diffusivity of the ground. Strictly this conclusion may only be drawn for non-spreading pools, but this is the situation where three-dimensional effects will be most important. We shall assume, on dimensional grounds, that this criterion is also valid for spreading pools. The models of Sections 4.4 and 4.3 indicate $a \sim 1/4$ and $a \sim 1/\ell n(eR\tilde{h}/2K)$ respectively. The second value may be rather smaller than the first, and so there is some uncertainty over the best value of 'a'. However for typical ground, α is sufficiently small (roughly 10^{-6} m^2/s for concrete) that horizontal conduction effects may be considered negligible except for very small, bunded pools or relatively non-volatile liquids, or pools which have become small (or broken up) due to evaporation and surface tension.

5.2 Thermal properties of the ground

Even when a theory of heat conduction under a pool is complete there remains the question of the values of the thermal conductivity K, thermal diffusivity α, and the heat transfer coefficients. In particular, in Reference 5, it is found that the value of K needed to fit the theory to evaporation experiments is rather larger (typically by a factor of 3 or 4) than that found more directly from conduction experiments. It is also found that the values of K and $\tilde{h}$ which fit the vaporisation rate do not necessarily fit the temperature profile in the ground. The authors of that work put forward the idea that K may depend significantly on temperature, but this is not clearly demonstrated. However further evidence of this is given by other authors – see below.

Shaw and Briscoe[1] note that earlier vaporsation experiments also give apparently large values for K, and comment that surface roughness may contribute to this effect. However experience with the surface heat transfer coefficient suggests that surface features with a length-scale of order λ should only affect the heat flux for times up to order (λ^2/α). For the effect to last as long as an hour (the time-scale of the experiments of Reference 5), then one would need (for concrete as discussed in Section 4.3) surface roughness on a scale of about 7 cm, which is rather too large to simply be considered as 'roughness'. Surface roughness on a scale smaller than h^{-1} in the models of Sections 2.4 and 4 would presumably affect the optimum value of h but little else.

More recently Moorhouse and Carpenter[11] have also noted that the conduction parameters (in particular $K/\sqrt{\alpha}$ which determines the heat flux into a boiling, bunded pool) will depend in general on temperature. They suggest that taking some empirical average value is a useful procedure. They show that for spills of LNG (boiling point 112°K) on limestone chippings that the thermal properties at 116°K give better results than those at ambient temperature. The simplest way of obtaining the best empirical value of $K/\sqrt{\alpha}$ for a safety analysis of a fixed installation storing a given liquid is, they suggest, to fit the theory to the results of experiments done with that liquid on the given surface around the storage tank. In view of the uncertainties found by other authors, this is a valid viewpoint, but is not so easily applicable to transport hazards where assorted liquids may be spilt on various types of terrain.

The temperature dependence of the thermal properties of the ground is therefore the most promising explanation of the anomalously high values of K found by some authors. If this is indeed the correct explanation, these experiments with different liquids on different surfaces may provide the best way of obtaining optimum values for conduction parameters for use in safety analyses.

5.3 Conclusions

For many applications Equation (3.14) (with the approximation $\tilde{A}(t, \tau) \approx A(\tau)$) should provide an adequate estimate of the heat flux conducted into a spreading pool of varying temperature. If the heat transfer coefficient $\tilde{h}$ is large enough (or alternatively if the vaporisation continues for a period much longer than $t_0 = K^2/(\tilde{h}^2\alpha)$) then the simpler result (2.15) will be equally good. These results are more generally applicable than earlier work (e.g. References 1, 5). There are two caveats. Firstly, this simple result is valid only if three-dimensional effects are unimportant. This criterion is summarised in Section 5.1. In fact in most safety analyses three-dimensional conduction effects are not expected to be significant. A simple prescription for estimating and allowing for such effects is given in Section 4.4. Secondly, the values of the conductivity K and diffusivity α should be taken to fit vaporisation experiments. These, probably because of temperature dependence effects, indicate rather larger conductivity than would otherwise be expected.[5]

Finally, the conducted heat flux given by (2.15) or (3.14) will be one of the elements specifying the evolution of the temperature of the pool. This means that the integral which remains must be done at each time step in any numerical solution. It is therefore important to be able to do it quickly and efficiently. We show how this may be achieved in Appendix D.

6. REFERENCES

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7. NOMENCLATURE

$a(t, \tau)$	a function defined in Appendix D for use in the numerical computation of conduction integrals
$A(t)$	the pool area
$\tilde{A}(t, \tau)$	a function of the pool evolution defined in (2.13)
$C(t, \tau)$	the 'horizontal conduction factor' defined in (4.17)
erfc(x)	the complementary error function
$\overline{\mathrm{erfc}}(x)$	e^{x^2} erfc(x)
$G_V G_V' G_1 G_2 G_H$	Green's functions (see Appendix B and (4.4))
$\tilde{h}$	the surface heat transfer coefficient
h	$= \tilde{h}/K$
$I_n(x)$	the modified Bessel function of the first kind
$\tilde{I}_n(x)$	$= e^{-x} I_n(x)$
K	the thermal conductivity of the ground
$P(t)$	the set of surface points covered by the pool at time t
$Q(t, z, \vec{x})$	the vertical heat flux density in the ground (defined positive for upward heat flux)
Q_T	the total heat flux into the pool
R	the radius of a circular pool
$S(t, z, \vec{x})$	an auxiliary field defined in (3.3)
t	time
t_0	the surface heat-transfer timescale (4.19)
t_H	the horizontal heat-transfer timescale (4.20)
$\vec{x}$	the horizontal 2-vector co-ordinate
z	the vertical (depth) co-ordinate
α	the thermal diffusivity of the ground
$\Phi(t, \vec{x})$	the surface temperature
$\phi(t)$	the average pool temperature

8. APPENDIX A

Heat conduction at the surface

The conducted heat flux density at the point $(\vec{x}, z)$ in the model of Section 2 is

$$q(t, z, \vec{x}) = \frac{K}{\sqrt{4\pi\alpha}} \int_0^t d\tau\,(t-\tau)^{-3/2} \left[1 - \frac{2z^2}{4\alpha(t-\tau)}\right] e^{\frac{z^2}{4\alpha(t-\tau)}}\,\Phi(\tau, \vec{x}) \qquad \text{... (A.1)}$$

To obtain the surface heat flux the limit $z \to 0$ must be taken carefully. This can be done by using the identity

$$\Phi(\tau, \vec{x}) = \Phi(t, \vec{x}) - \left[\Phi(t, \vec{x}) - \Phi(\tau, \vec{x})\right] \qquad \text{... (A.2)}$$

to find

$$q(t, 0, \vec{x}) = \lim_{z \to 0} \frac{K\Phi(t, \vec{x})}{\sqrt{4\pi\alpha}} \int_0^t d\tau\,(t-\tau)^{-3/2} \left[1 - \frac{2z^2}{4\alpha(t-\tau)}\right] e^{-\frac{z^2}{4\alpha(t-\tau)}}$$

$$- \frac{K}{\sqrt{4\pi\alpha}} \int_0^t d\tau\,(t-\tau)^{-3/2} \left[\Phi(t, \vec{x}) - \Phi(\tau, x)\right] \qquad \text{... (A.3)}$$

In the second term we have simply set $z = 0$, as the zero in $\Phi(t, \vec{x}) - \Phi(\tau, \vec{x})$ as $\tau \to t$ softens the singularity in the integrand sufficiently. The more delicate limit in the first term may now be taken as follows:

A simple substitution recasts the integral as

$$\lim_{z \to 0} 4 \left(\frac{\alpha}{z^2}\right)^{1/2} \int_{\frac{z}{\sqrt{4\alpha t}}}^{\infty} dx\,(1 - 2x^2)\,e^{-x^2} \qquad \text{... (A.4)}$$

Using the identity

$$\int_0^{\infty} dx\,(1 - 2x^2)\,e^{-x^2} = 0 \qquad \text{... (A.5)}$$

the expression (A.4) becomes

$$- \lim_{z \to 0} 4 \left(\frac{\alpha}{z^2}\right)^{1/2} \int_0^{\frac{z}{\sqrt{4\alpha t}}} dx\,(1 - 2x^2)\,e^{-x^2} \qquad \text{... (A.6)}$$

which is straightforwardly evaluated by expanding the integrand in powers of x^2. The result is $-2t^{-1/2}$.

The surface heat flux density (A.3) thus becomes

$$q(t, 0, \vec{x}) = \frac{-K\Phi(t, \vec{x})}{\sqrt{\pi\alpha t}} - \frac{K}{\sqrt{4\pi\alpha}} \int_0^t d\tau\,(t-\tau)^{-3/2} \left[\Phi(t, \vec{x}) - \Phi(\tau, \vec{x})\right] \qquad \text{... (A.7)}$$

as given in (14).

9. APPENDIX B

Green's functions for vertical conduction

The Green's function

$$G_v(z,t) = \frac{z}{(4\pi\alpha t^3)^{1/2}} \exp\left(-\frac{z^2}{4\alpha t}\right) \qquad \ldots (B.1)$$

which was derived in Section 2.2, may conveniently be rewritten as

$$G_v(z,t) = \frac{\partial}{\partial t} \operatorname{erfc}\left(\frac{z}{(4\alpha t)^{1/2}}\right) \qquad \ldots (B.2)$$

and hence

$$G_v'(z,t) \equiv \frac{\partial G_v(z,t)}{\partial z} = -\frac{\partial}{\partial t}\left(\frac{1}{(\pi\alpha t)^{1/2}} e^{-\frac{z^2}{4\alpha t}}\right) \qquad \ldots (B.3)$$

Equation (B.2) also leads to relatively simple expressions for the Green's functions

$$G_1(z,t) = h \int_z^\infty dz' \, e^{h(z-z')} G_v(z',t) \qquad \ldots (B.4)$$

and

$$G_2 = h\left(G_1 - G_v\right) \qquad \ldots (B.5)$$

which were introduced in Section 3. These are

$$G_1(z,t) = \frac{\partial}{\partial t}\left\{\exp\left(\frac{-z^2}{4\alpha t}\right)\left[\overline{\operatorname{erfc}}\left(\frac{z}{(4\alpha t)^{1/2}}\right) - \overline{\operatorname{erfc}}\left(\frac{z}{(4\alpha t)^{1/2}} + (h^2\alpha t)^{1/2}\right)\right]\right\} \qquad \ldots (B.6)$$

$$G_2(z,t) = -h\frac{\partial}{\partial t}\left\{\exp\left(\frac{-z^2}{4\alpha t}\right)\overline{\operatorname{erfc}}\left(\frac{z}{(4\alpha t)^{1/2}} + (h^2\alpha t)^{1/2}\right)\right\} \qquad \ldots (B.7)$$

where $\overline{\operatorname{erfc}}$ is defined in Section 1.4.

The following properties of G_2, used in the text, are easily found from (B.6):

$$G_2(0,t) = (\alpha t_0^3)^{-1/2}\left\{\overline{\operatorname{erfc}}\left(\left(\frac{t}{t_0}\right)^{1/2}\right) - \frac{1}{\sqrt{\pi}}\left(\frac{t}{t_0}\right)^{-1/2}\right\} \qquad \ldots (B.8)$$

$$\lim_{z\to 0} \int_0^t d\tau \, G_2(z, t-\tau) = (\alpha t_0)^{-1/2} \, \overline{\operatorname{erfc}}\left(\left(\frac{t}{t_0}\right)^{1/2}\right) \qquad \ldots (B.9)$$

where

$$t_0 \equiv (h^2\alpha)^{-1}$$

For the general expressions (B.6) and (B.7) it is straightforward to perform the derivative on t but it does not simplify the expressions.

10. APPENDIX C

The horizontal conduction factor

10.1 The horizontal conduction factor for a concentrically spreading circular pool

The horizontal conduction factor $C(t, \tau)$ defined in Section 3 reduces, in the case of a concentrically spreading circular pool of radius $R(t)$, to

$$C(t, \tau) = \frac{1}{4\pi\alpha(t-\tau)} \int_{|\vec{x}| \leqslant R(t)} d^2\vec{x} \int_{|\vec{y}| \leqslant R(\tau)} d^2\vec{y} \exp\left(\frac{-(\vec{x}-\vec{y})^2}{4\alpha(t-\tau)}\right) \quad \text{... (C.1)}$$

This is conveniently rewritten as

$$C(t, \tau) = 4\pi\alpha(t-\tau) \; H\left(\frac{R^2(t)}{4\alpha(t-\tau)}, \frac{R^2(\tau)}{4\alpha(t-\tau)}\right) \quad \text{... (C.2)}$$

with

$$H(X, Y) \equiv \frac{1}{\pi^2} \int_{|\vec{x}| \leqslant \sqrt{X}} d^2\vec{x} \int_{|\vec{y}| \leqslant \sqrt{Y}} d^2\vec{y} \; e^{-(\vec{x}-\vec{y})^2} \quad \text{... (C.3)}$$

The fourfold integral (C.3) may be simplified in various ways. We shall give a few which preserve the manifest X-Y symmetry.

If the vectors $\vec{x}$ and $\vec{y}$ are written in polar co-ordinates, the angular integrals can be done straightforwardly to give

$$H(X, Y) = \int_0^X d\lambda \int_0^Y dw \; e^{-(\lambda+w)} \, I_0(2\sqrt{\lambda w}) \quad \text{... (C.4)}$$

where I_0 is the modified Bessel function defined in (2).

The double Laplace transform of $H(X, Y)$,

$$\tilde{H}(p, q) = \int_0^\infty dX \, e^{-pX} \int_0^\infty dY e^{-qY} \, H(X, Y) \quad \text{... (C.5)}$$

is tantalisingly simple:

$$\tilde{H}(p, q) = \frac{1}{pq\,(p+q+pq)} \quad \text{... (C.6)}$$

This may be used to obtain two more expressions for $H(X, Y)$.

Firstly, rewriting (C.6) as

$$\tilde{H}(p, q) = \frac{1}{pq} \sum_{n=0}^{\infty} \frac{1}{(p+1)^n} \, \frac{1}{(q+1)^n} \quad \text{... (C.7)}$$

(which converges for $|p+1|, |q+1| > 1$) gives the inverse transform in the form

$$H(X,Y) = \sum_{n=0}^{\infty} P(n+1,X)\,P(n+1,Y) \qquad \text{... (C.8)}$$

where

$$P(n+1,X) = \frac{1}{n!}\int_{o}^{X} dx\, x^{n}\, e^{-x} \qquad \text{... (C.9)}$$

or

$$P(n+1,X) = 1 - \left[e^{-X} \sum_{m=0}^{n} \frac{X^{m}}{m!} \right] \qquad \text{... (C.10)}$$

The series representation (C.8) converges most rapidly if X or Y is small. For large X, P (n, X) is of order one for $n \ll X$ and falls off rapidly with n as n becomes much larger than X. Thus (C.8) implies that H (X, Y) is approximately Min (X, Y) for large X and Y.

Secondly, the Laplace transform (C.6) may be manipulated to give

$$\widetilde{H}(p,q) = \frac{1}{pq\,(p+q)} - \frac{1}{(p+q+pq)(p+q)} \qquad \text{... (C.11)}$$

The first term of (C.11) is the transform of Min (X, Y). The second term of (C.11) may be treated in a symmetric way using the integral representation

$$(p+q)^{-1} = \int_{o}^{\infty} d\phi\; e^{-p\phi}\, e^{-q\phi} \qquad \text{... (C.12)}$$

The Laplace transforms may now be inverted to give

$$H(X,Y) = \mathrm{Min}(X,Y) - f(X,Y) \qquad \text{... (C.13)}$$

with

$$f(X,Y) = \int_{o}^{\mathrm{Min}(X,Y)} d\phi\; e^{-(X-\phi)}\, e^{(Y-\phi)}\, I_0\left(2\sqrt{(X-\phi)(Y-\phi)}\right) \qquad \text{... (C.14)}$$

A suitable change of variables recasts (C.14) as

$$f(X,Y) = \int_{o}^{XY} d\xi\; \frac{e^{-\sqrt{(X-Y)^2+4\xi}}}{\sqrt{(X-Y)^2+4\xi}}\; I_0\left(2\sqrt{\xi}\right) \qquad \text{... (C.15)}$$

or as

$$f(X,Y) = \frac{1}{2}\int_{|X-Y|}^{(X+Y)} dw\; e^{-w}\, I_0\left(\sqrt{w^2-(X-Y)^2}\right) \qquad \text{... (C.16)}$$

In terms of this function, C (t, τ) is given by

$$C(t,\tau) = \mathrm{Min}\left(\pi R^2(t),\, \pi R^2(\tau)\right) + 4\pi\alpha\,(t-\tau)\, f\left(\frac{R^2(t)}{4\alpha\,(t-\tau)},\, \frac{R^2(\tau)}{4\alpha\,(t-\tau)}\right) \qquad \text{... (C.17)}$$

In the limit where horizontal conduction effects are unimportant ($\alpha \to 0$), C reduces to Min ($\pi R^2(t)$, $\pi R^2(\tau)$) which is directly comparable with (18). For an outward spreading pool it is $\pi R^2(t)$.

10.2 Horizontal conduction effects under a pool of constant radius

For a pool of constant radius R the function C (t, τ) reduces to

$$C_B(t - \tau) = 4\pi\alpha (t - \tau)\ H \left(\frac{R^2}{4\alpha(t - \tau)}\ ,\ \frac{R^2}{4\alpha(t - \tau)} \right) \qquad \text{... (C.18)}$$

with

$$H(X, X) = X - f(X, X) \qquad \text{... (C.19)}$$

For Y = X the integral (B.17) yields

$$f(X, X) = X \left(\tilde{I}_0(2X) + \tilde{I}_1(2X) \right) \qquad \text{... (C.20)}$$

and hence

$$C_B(t - \tau) = \pi R^2 \left\{ 1 - \tilde{I}_0 \left(\frac{R^2}{2\alpha(t - \tau)} \right) - \tilde{I}_1 \left(\frac{R^2}{2\alpha(t - \tau)} \right) \right\} \qquad \text{... (C.21)}$$

where $\tilde{I}_n$ is defined in Equation (1.4).

The properties of the modified Bessel functions imply that

$$\frac{\partial}{\partial \tau} C_B(t - \tau) = \frac{\pi R^2}{(t - \tau)}\ \tilde{I}_1 \left(\frac{R^2}{2\alpha(t - \tau)} \right) \qquad \text{... (C.22)}$$

This expression is used in Section 3.2.

11. APPENDIX D

Numerical considerations

In any numerical calculation of the heat flux into a pool at least one integral over the past history of the pool must be done. This must be done at each time step in a program to solve the differential equation (schematically) of the form

$$\frac{d\phi}{dt} \sim Q_T(t) \qquad \text{... (D.1)}$$

for the temperature of the pool. Thus accuracy and efficiency are crucial. Let us illustrate how this can be done using the model result (2.14). First let us rewrite this equation as

$$Q_T(t) = \frac{KA(t)}{\sqrt{\pi\alpha}} \left\{Q_0 + Q_1 + Q_2 + Q_3\right\} \qquad \text{... (D.2)}$$

with

$$Q_0 \equiv \frac{-\phi(t)}{\sqrt{t}} \qquad \text{... (D.3)}$$

$$Q_1(t) \equiv -\frac{1}{2}\int_0^t d\tau (t-\tau)^{-3/2} \left[\frac{Q(t)\dot{A}(t)}{A(t)} (t-\tau)\right] \qquad \text{... (D.4)}$$

$$Q_2(t) \equiv \frac{1}{2}\int_0^t d\tau (t-\tau)^{-3/2} \left[\frac{\phi(t)\ddot{A}(t)}{A(t)} \frac{(t-\tau)^2}{2!}\right] \qquad \text{... (D.5)}$$

leaving

$$Q_3(t) = \frac{1}{2}\int_0^t d\tau\, (t-\tau)^{-3/2} \left[\phi(t)a(t,\tau) - \frac{[\phi(t)-\phi(\tau)]\,A(\tau)}{A(t)}\right] \qquad \text{... (D.6)}$$

where

$$a\,(t,\tau) \equiv \left[\frac{A(\tau)}{A(t)} - 1 + (t-\tau)\frac{\dot{A}(t)}{A(t)} - \frac{1}{2}(t-\tau)^2\frac{\ddot{A}(t)}{A(t)}\right] \qquad \text{... (D.7)}$$

Note that $a\,(t,\tau)$ vanishes as $(t-\tau)^3$ as $\tau \to t$.

Now we find

$$Q_1 = -\frac{\phi(t)}{\sqrt{t}} \cdot \left[\frac{t\dot{A}(t)}{A(t)}\right] \qquad \text{... (D.8)}$$

and

$$Q_2 = \frac{\phi(t)}{\sqrt{t}} \left[\frac{t^2\ddot{A}(t)}{6A(t)}\right] \qquad \text{... (D.9)}$$

and only Q_3 must be evaluated numerically. In many circumstances Q_3 will be a small contribution to the total. For example if A(t) is constant, linear, or quadratic in t then $a\,(t,\tau) = 0$, and the first contribution to (D.6) vanishes. Also, if the temperature ϕ varies slowly in time then the second contribution to (D.6) is small. Thus the total heat flux can be estimated accurately without an unduly time-consuming evaluation of Q_3. For numerical purposes (D.6) is conveniently rewritten as

$$Q_3(t) = \frac{\phi(t)}{\sqrt{t}} \int_o^l \frac{d\xi}{\xi^2} \left\{ a\left(t, t(1-\xi^2)\right) - \left[1 - \frac{\phi\left(t(1-\xi^2)\right)}{\phi(t)} \right] \frac{A\left(t(1-\xi^2)\right)}{A(t)} \right\} \qquad \text{... (D.10)}$$

noting that $a \underset{\xi \to o}{\sim} \xi^6$ and $\left[\phi\left(t(1-\xi^2)\right)/\phi(t)\right] \underset{\xi \to o}{\to} 1 - O(\xi^2)$.

Similar manipulations can be performed to advantage on other models for $Q_T(t)$. In the model of Section 3, for example, the corresponding results are

$$Q_0 = \frac{-\phi(t)}{\sqrt{t}} \left(\frac{t}{t_0}\right)^{1/2} \pi^{1/2} \, \overline{\text{erfc}} \left(\left(\frac{t}{t_0}\right)^{1/2}\right) \qquad \text{... (D.11)}$$

$$Q_1 = \frac{-\phi(t)}{\sqrt{t}} \left[\frac{t\dot{A}(t)}{A(t)}\right] . \; 2 \; F_3 \left(\frac{t}{t_0}\right) \qquad \text{... (D.12)}$$

$$Q_2 = \frac{+\phi(t)}{\sqrt{t}} \left[\frac{t^2\ddot{A}(t)}{2A(t)}\right] 2 \; F_5 \left(\frac{t}{t_0}\right) \qquad \text{... (D.13)}$$

$$Q_3 = \frac{\phi(t)}{\sqrt{t}} \, 2 \int_o^l d\xi \, . \left\{1 - \xi \sqrt{\pi} \left(\frac{t}{t_0}\right)^{1/2} \overline{\text{erfc}} \left(\xi \left(\frac{t}{t_0}\right)^{1/2}\right)\right\} . \left(\frac{t}{t_0}\right)$$

$$. \left\{ a\left(t, t(1-\xi^2)\right) - \left[1 - \frac{\phi\left(t(1-\xi^2)\right)}{\phi(t)} \right] \frac{A\left(t(1-\xi^2)\right)}{A(t)} \right\} \qquad \text{... (D.14)}$$

where

$$F_m(x) = x^{1/2(2-m)} \int_o^{\sqrt{x}} dy \left\{y^{m-1} - \sqrt{\pi} \, y^m \, \overline{\text{erfc}} \, (y)\right\} \qquad \text{... (D.15)}$$

These integrals may be calculated straightforwardly for odd integers m by writing

$$F_m(x) = x^{1/2(2-m)} \left\{L_m(\sqrt{x}) - L_m(o)\right\} \qquad \text{... (D.16)}$$

where

$$L_1(y) = -\frac{1}{2} \sqrt{\pi} \, \overline{\text{erfc}} \, (y) \qquad \text{... (D.17)}$$

and

$$L_{m+1}(y) = -\frac{m}{2} L_{m-1}(y) + \frac{m}{2(m-1)} y^{m-1} - \frac{1}{2} \sqrt{\pi} \, y^m \, \overline{\text{erfc}} \, (y) \qquad \text{... (D.18)}$$

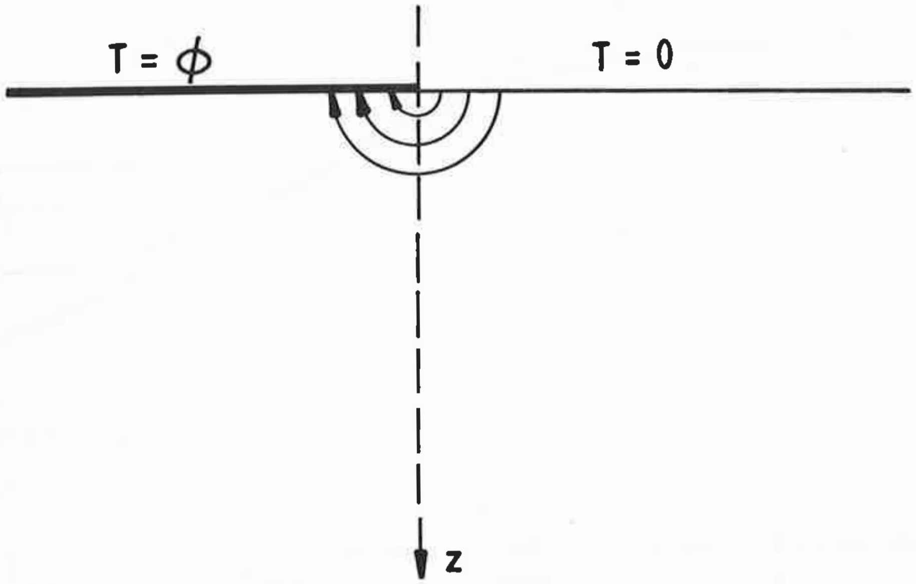

FIGURE 1

THE SURFACE TEMPERATURE DISTRIBUTION DESCRIBED IN SECTION 4.2 THE LARGE TEMPERATURE GRADIENT NEAR THE ORIGIN CAUSES AN INFINITE HEAT FLUX INDICATED (FOR $\phi < 0$) BY THE ARROWS

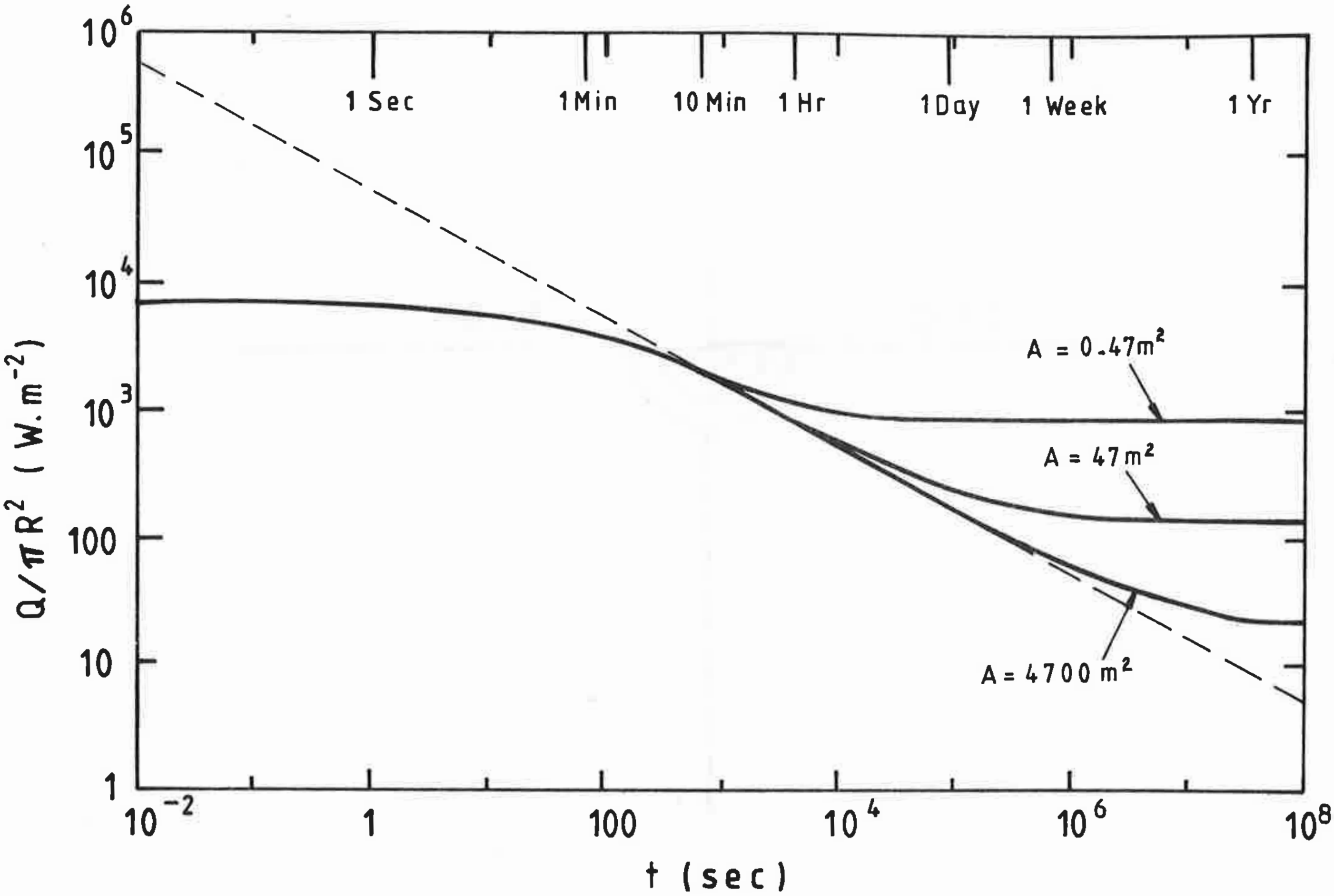

FIGURE 2

THE AREA-AVERAGED HEAT FLUX DENSITY INTO THE POOLS DESCRIBED IN SECTION 4.3. THE BROKEN LINE GIVES THE VERTICAL CONDUCTION APPROXIMATION OF SECTION 2.3. THE THREE SOLID LINES GIVE THE PREDICTIONS OF THE MODEL OF SECTION 4. THE RESULT AT SMALL TIME DUE TO IMPERFECT THERMAL CONTACT IS INDEPENDENT OF POOL SIZE. THE TIME AT WHICH HORIZONTAL CONDUCTION EFFECTS BECOME IMPORTANT DEPENDS STRONGLY ON THE POOL SIZE. THE MODEL OF REF. 5 FOLLOWS THE SOLID LINE AT SMALL t AND THE BROKEN LINE AT LARGE t.

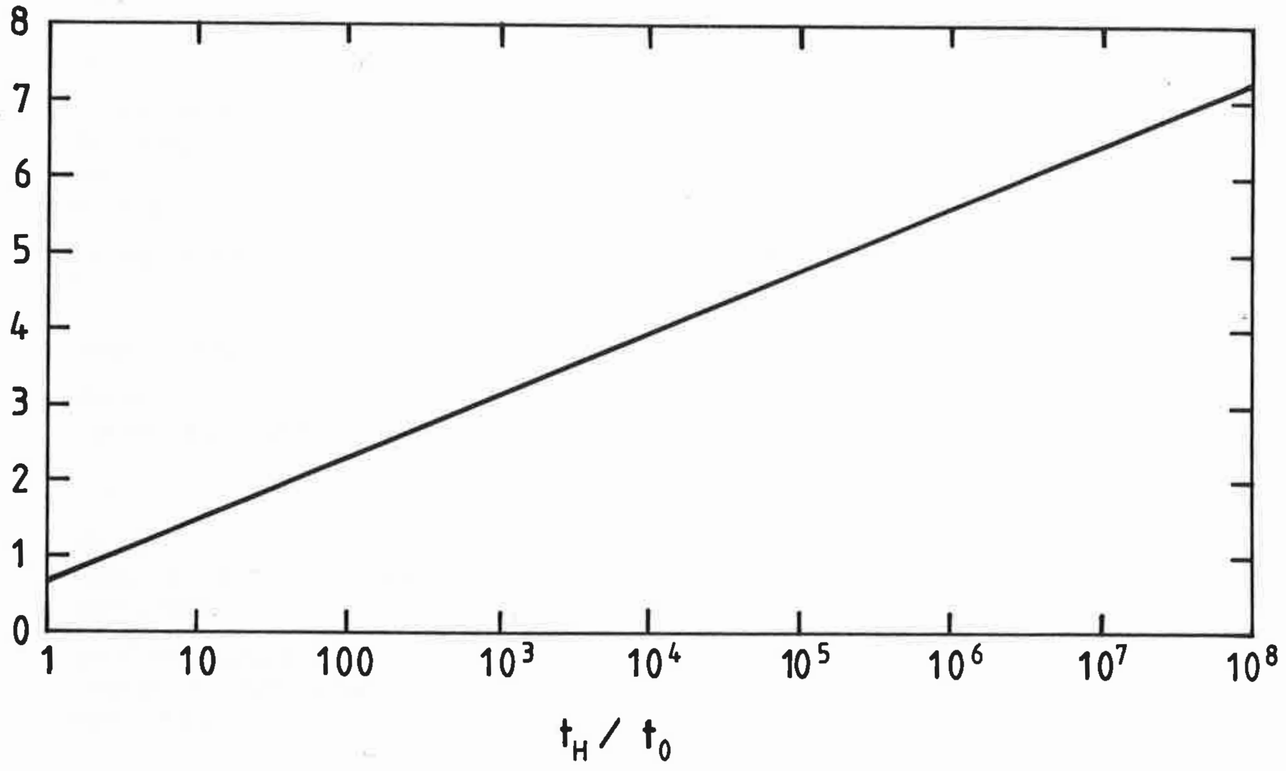

FIGURE 3

THE RATIO OF THE ASYMPTOTIC, STEADY-STATE HEAT FLUXES PREDICTED BY THE MODELS OF SECTIONS 4 AND 5.2. THE HORIZONTAL AXIS IS $t_H / t_0 = (Rh/2)^2$ AS DEFINED IN SECTION 4

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