Uncertainty and Sensitivity Analysis for Complex Simulation Models

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- f may be deterministic.
- Computer experiment: evaluating f at difference choices of x
  - A 'model run': evaluating f at a single choice of x.

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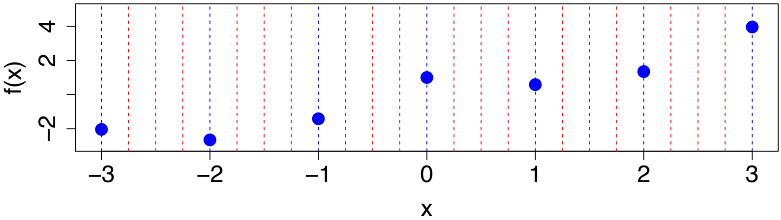
- What do we do if f is computationally expensive?

Computationally expensive models



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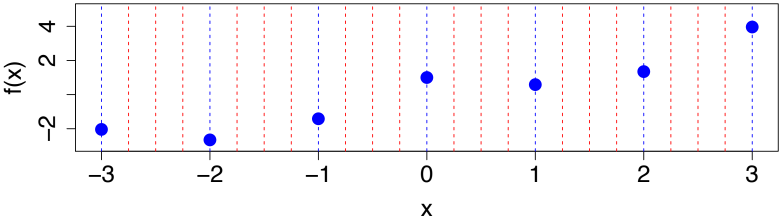

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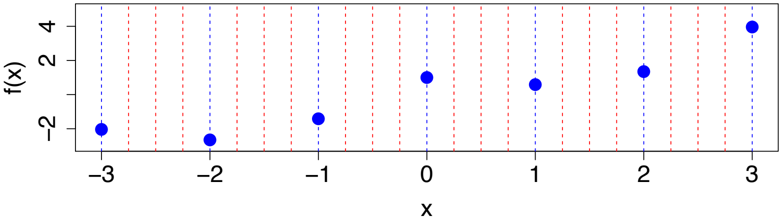

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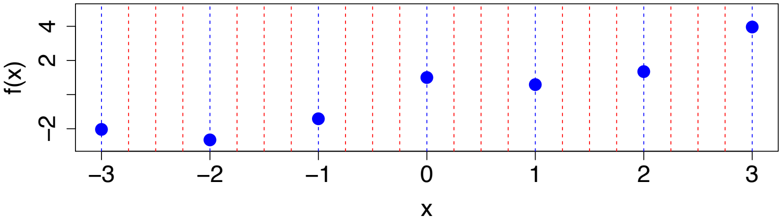

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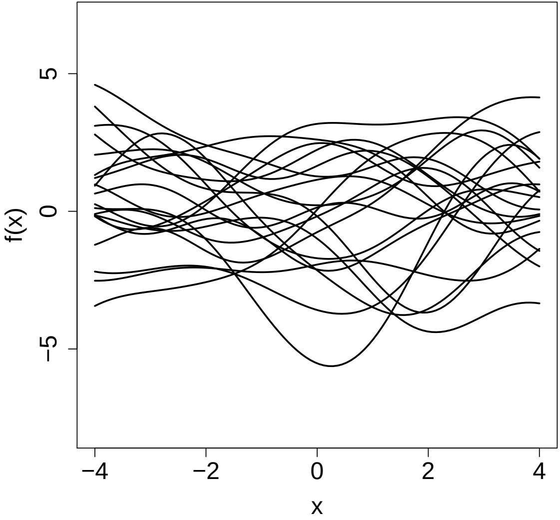
- Popular technique: Gaussian process emulation (Sacks et al 1989)

Gaussian process emulators



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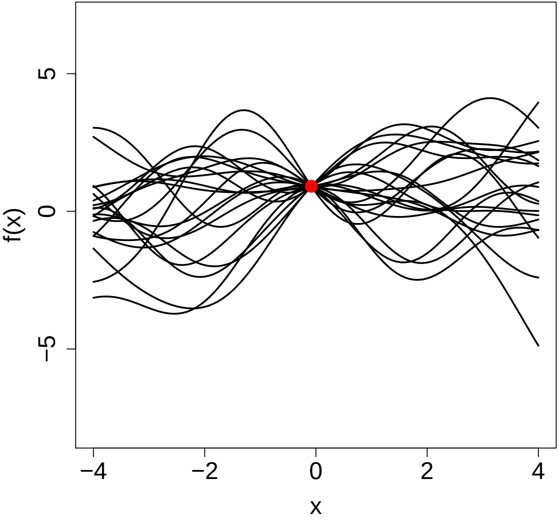

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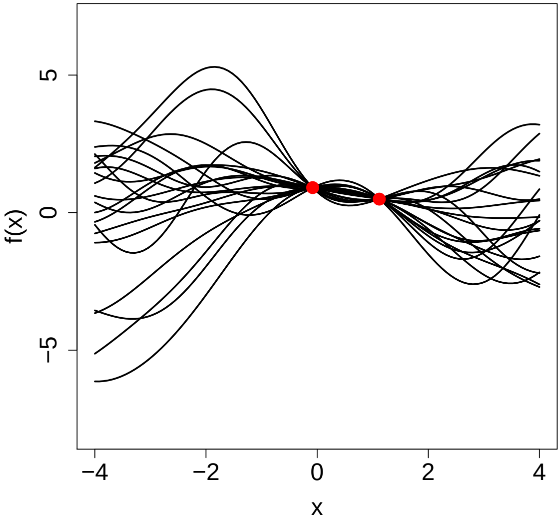

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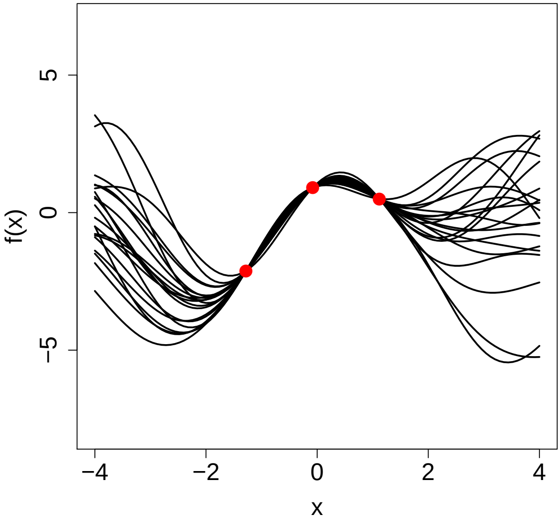

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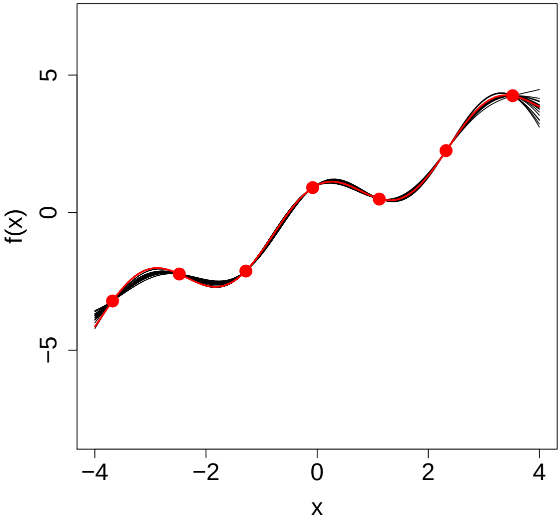

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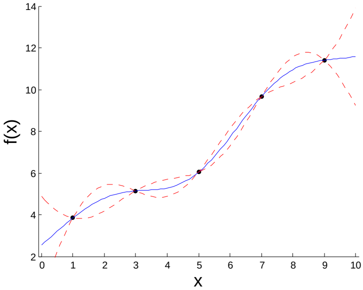
- If the simulator has given us the value of $f(x)$, the emulator will give us the *same* value



Example: 18 input climate model, 255 model runs

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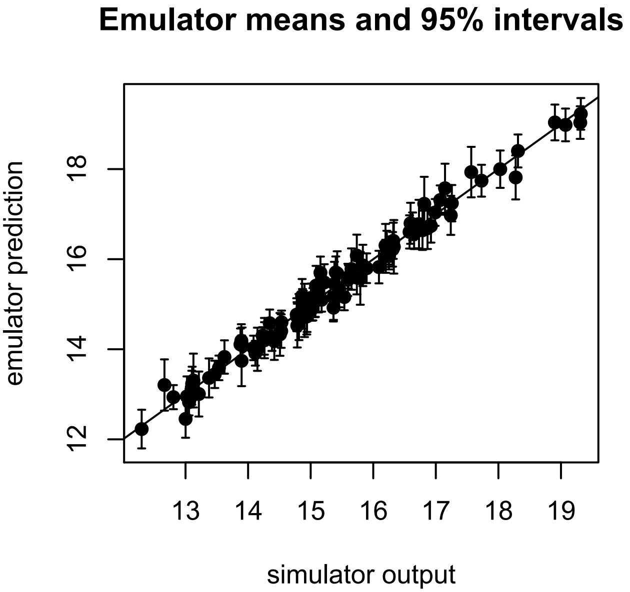

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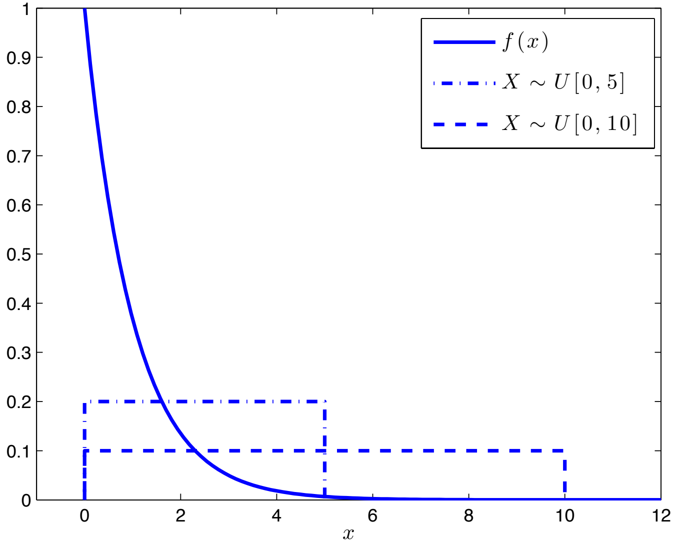
Increasing b increases the variance of X but *decreases* the variance of Y



1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
2
4
6
8
10
12
x
$f(x)$
$X \sim U[0, 5]$
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- Sensitivity analysis: which elements in $\boldsymbol{X} = \{X_1, \ldots, X_d\}$ are most responsible for the uncertainty in $Y = f(\boldsymbol{X})$?
- Depends on both f *and* $p_{\boldsymbol{X}}$
  - Typically, cannot quantify 'importance' of input by studying f alone

Variance based sensitivity analysis

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- Can speed up computation with emulator (Oakley & O'Hagan, 2004)

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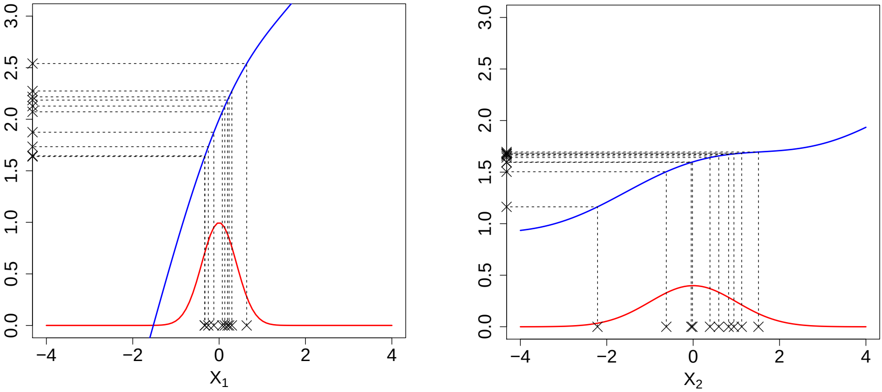

— $p_{X_1}(x_1)$

— $E(Y|X_1 = x_1)$

— $p_{X_2}(x_2)$

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Example[1]: modelling Rotavirus

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- GSK analysis investigated sensitivity of output to 9 inputs, using 8200 model runs
- We consider sensitivity of output to 20 inputs, using 340 model runs

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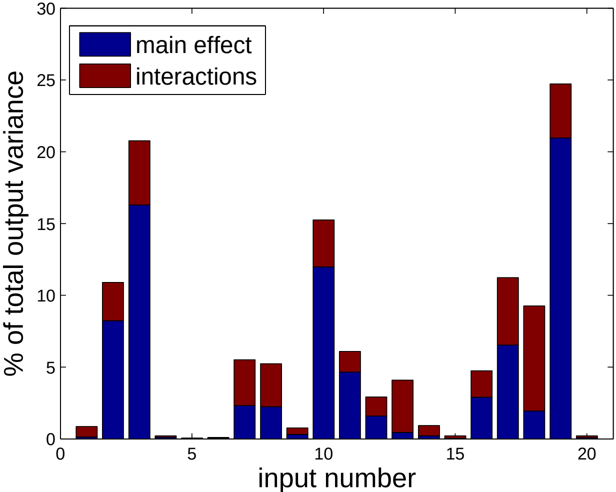
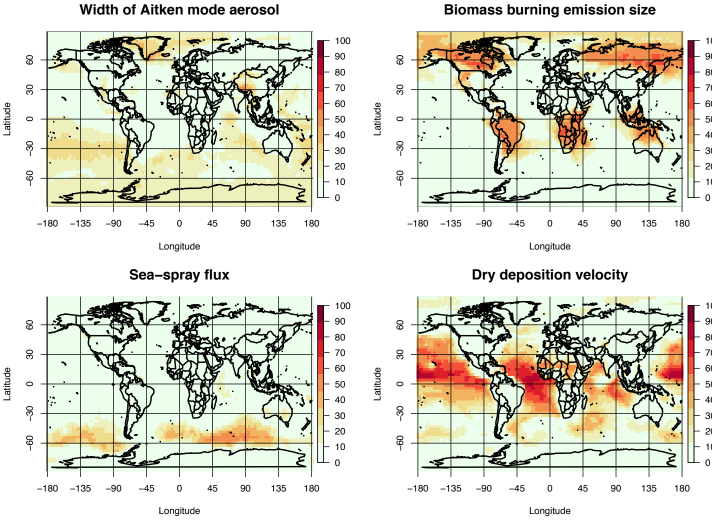
Variance based sensitivity analysis

Analysis for an individual output: no. of infections in 2-3 age group after 2 years



SA for a global aerosol model: Lee et al. 2013



Acknowledgement: figure kindly provided by Lindsay Lee

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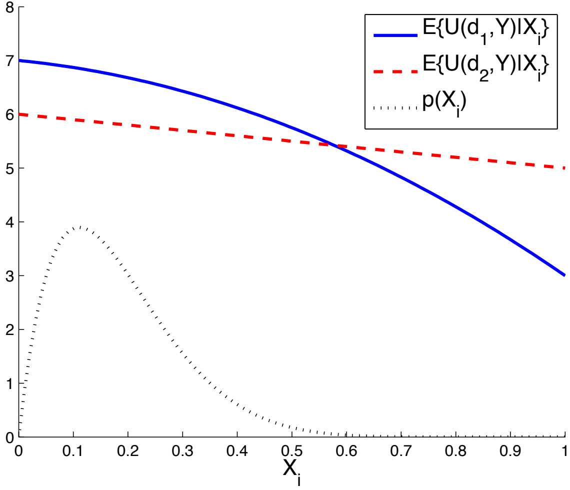
- Expected value of learning X_i before making decision is

$$E_{X_i}\left[\max_a E\{U(a, Y)|X_i\}\right] - U^*,$$

the **partial EVPI** of X_i (**Expected Value of Perfect Information**)

Sensitivity analysis for decision making



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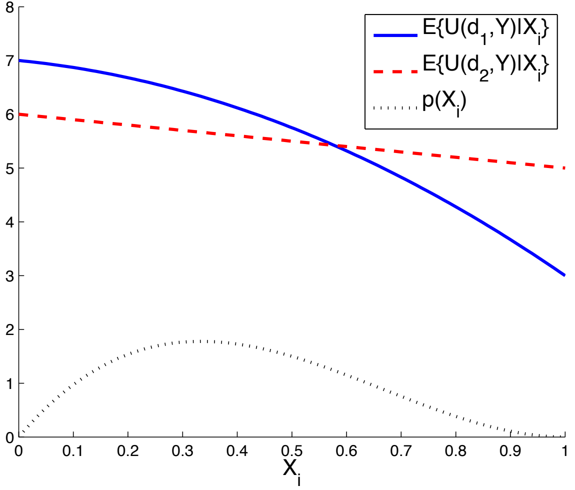
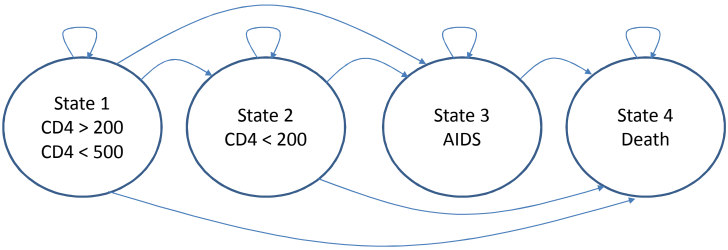


Case study: health economic modelling (Strong and Oakley, 2014)



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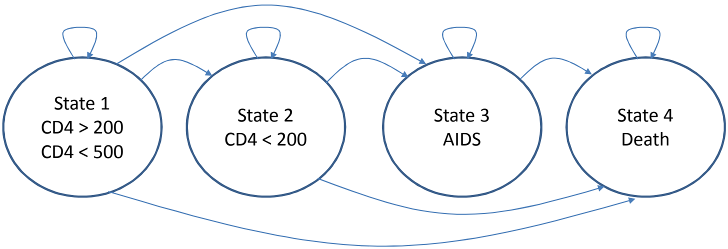

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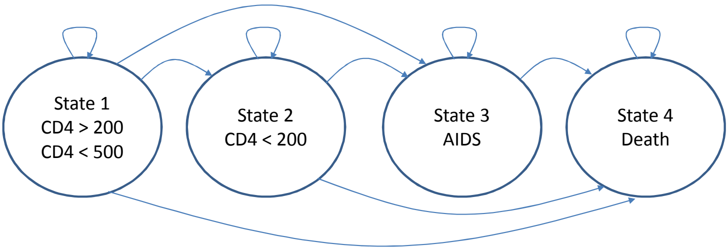

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- Consider extra parameters to represent *model structure* uncertainty

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Parameter	Partial EVPI			
	Base case	Scenario 1	Scenario 2	Scenario 3
Transition probabilities	£0	£0	£0	£1.17
Relative risk	£169.91	£193.09	£64.63	£164.55
Costs	£194.41	£201.72	£65.17	£167.53
Discrepancy terms	-	£7.86	£110.21	£699.06
Overall EVPI	£365.42	£401.53	£333.43	£957.28

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    - EVPI for more focussed decision-making context

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- Sensitivity analysis tools for investigating input uncertainty.
  - variance-based approach as a 'general purpose' method
  - EVPI for more focussed decision-making context
- Careful specification of input uncertainty is important!

References

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